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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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finite number. In addition to the two groups of order 24 noted above there are 13 others, which were first completely determined in 1896 by the present writer. The lowest order for which the number of groups exceeds the order is 32. There are 51 distinct groups of this order.

The verification of several of these statements would carry us beyond easy group theory. They may serve, however, to exhibit a type of inquiry relating to our subject. Fortunately, some of the most important and far-reaching phases of this subject are also the easiest. For instance, the group of all the transformations of space which leave invariant the distances between every pair of points can easily be comprehended. Those geometric figures which can be transformed into each other under this group may be called equivalent and we may confine our attention to the study of geometric properties which remain invariant under the transformation of this group. We thus obtain a body of knowledge commonly called *Euclidean Geometry*, but this term is also often used with different meanings. Following the custom introduced by Gauss some still use it to denote all the geometry in which the parallel postulate is assumed.

It is clear that in geometry it is undesirable to endeavour to study every figure as an individual since one could not make much progress in this way. What people have always done in this subject is to confine their attention to invariants under certain infinite groups of transformations. It is true that the ancients did not specify these groups and that we do not usually do this now in a first course in elementary geometry; but for the advanced student, at least, the developments become much clearer if this specification is explicitly made. If we add to the transformations noted above those which do not preserve the size of the figures but do preserve their angles, so that all similar figures are regarded as equivalent, we obtain a larger group, which has been called the principal group of geometry. The body of knowledge relating to the invariants of this group is commonly known as *Elementary Geometry*. In particular, all circles are equivalent in this geometry and all squares are also equivalent. Some writers call this geometry Euclidean Geometry. [See page 61, Volume 2, Pascal's *Repertorium der höheren Mathematik*, 1910.]

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On the Co-efficients in the expansion of cn (x, k).

By C. Krishnamachari and M. Bhimasena Rao.

Hancock in his *Theory of Elliptic Functions*, Vol. I, page 252, gives the first five co-efficients in the expansion of cn (x, k), as having been calculated by Gudermann in *Crelle*, Bd. XIX, p. 80. The authors pointed out in their paper: " *Determinants involving Specified Numbers*," Vol. XIV, No. 4, J. I. M. S., pp. 122—138, that the ascending Σ—table formed with $a_{2n-1} = (2n-1)^2$, $a_{2n} = (2n)^2k^2$ gives the co-efficients in the expansion of cn (x, k) and that with $a_{2n-1} = (2n-1)^2k^2$, $a_{2n} = (2n)^2$ gives the co-efficients in the expansion of dn (x, k)—*Ibid*, pp. 133, § 6. By way of illustration, we have shown below the calculation of the table for the first four co-efficients. With the help of the table, we have calculated five more co-efficients; any one interested may easily calculate the table as far as he desires. It is of course obvious that we can now also write down the ten co-efficients of dn (x, k).

Table.

1	$1+4k^2$	$1+4k^2$	$1+44k^2+16k^4$	$1+44k^2+16k^4$
$4k^2$	$10+4k^2$	$40k^2+16k^4$	$91+224k^2+16k^4$	$364k^2+896k^4+64k^6$
9	$10+20k^2$	$90+180k^2$		
$16k^2$				

$1+408k^2+912k^4+64k^6$

The *fifth* co-efficient, (also calculated by Gudermann), is

$$1+3688k^2 + 30768k^4 + 15808k^6+256k^8.$$

The sum of the co-efficients here is of course E_5 (Eulerian number)

We find that the *next five* co-efficients are as follows :—

the *sixth*: $1 + 33212k^2 + 870640k^4 + 1538560k^6 + 259328k^8 + 1024k^{10}$,

the *seventh*: $1 + 298932k^2 + 22945056k^4 + 106923008k^6$
$+ 65008896k^8 + 4180992k^{10} + 4096k^{12}$,

the *eighth*: $1 + 2690416k^2 + 586629984k^4 + 6337665152k^6$
$+ 9860488448k^8 + 2536974336k^{10} + 67047424k^{12}$
$+ 16384k^{14}$,

the *ninth*: $1 + 24213766k^2 + 14804306080k^4$
$+ 345558617984k^6 + 1165333452544k^8 + 782931974144k^{10}$
$+ 95153582080k^{12} + 1073463296k^{14} + 65536k^{16}$,

the *tenth*: $1 + 217924020k^2 + 3715483371744k^4$
$+ 17992189979904k^6 + 119641512257280k^8$
$+ 171748920960000k^{10} + 57102164668416k^{12}$
$+ 3497455190016k^{14} + 17178624000k^{16}$
$+ 262144k^{18}$.

An Extension of the Harmonic Relation between Conics.

BY R. VAIDYANATHASWAMY.

If there exists *one* self-polar triangle of a conic S inscribed in a second conic S′, then there exists an one-fold infinity and S′ is *out-polar* or *ex-harmonic** to S. On the other hand it is easily seen that there always exists a finite number of self-polar triangles of S with their three vertices situated severally on three assigned conics S_1, S_2, S_3. The object of the present note is to shew that under certain circumstances the conics S_1, S_2, S_3 admit an *infinity* of such triangles. For the symbolic method which is used throughout, reference may be made to Grace and Young: *Algebra of Invariants* (Ternary Forms).

1. The condition that three binary quadratics a_x^2, b_x^2, c_x^2 are in involution is

$$(bc)\,(ca)\,(ab) = 0.$$

Hence by Clebsch's Principle†, the envelope of lines cutting three conics a_x^2, b_x^2, c_x^2 in an involution range is

$$(bc\,\omega)\,(ca\,\omega)\,(ab\,\omega) = 0,$$

where the variables $\omega_1\ \omega_2\ \omega_3$ are contragredient to $x_1\ x_2\ x_3$. If the three conics in question have a common self-polar triangle whose vertices are $U_1 = 0$, $U_2 = 0$, $U_3 = 0$, it is evident from Geometry that U_1, U_2, U_3 should be factors of $(bc\,\omega)\,(ca\,\omega)\,(ab\,\omega)$; and from comparison of degrees we easily deduce that

$$(bc\,\omega)\,(ca\,\omega)\,(ab\,\omega) \equiv k\,(abc)^2\,U_1\,U_2\,U_3,$$

where k is numerical. If the three conics not only have a common self-polar triangle, but in addition belong to a four-point system—that is to say, if there is also a linear relation between a_x^2, b_x^2, c_x^2 — then again, it is obvious from Geometry that $(bc\,\omega)\,(ca\,\omega)\,(ab\,\omega)$ is zero identically; for a four-point system of conics cuts any line whatever in an involution

* The terms *out-polar* and *in-polar* have been suggested by Dr. Baker as equivalents of the German *stutzende* and *ruhende*. The terms *ex-harmonic* and *in-harmonic* have been used by myself in the same sense in previous communications to this Journal.

† Page 265 : Grace and Young.

range. The question arises whether in this last case we could obtain a contravariant of the three conics by subjecting the range of intersections to a single extra condition.

Let $S_1 = a_x^2 = a'^2_x = \ldots$, $S_2 = b_x^2$, $S_3 = \lambda\ S_1 + \mu\ S_2 = c_x^2$. Let a straight line cut the three conics in the points $P_1\ Q_1$, $P_2\ Q_2$, $P_3\ Q_3$ and let the single extra condition be that the harmonic conjugate of P_2 with respect to $P_1\ Q_1$ is *one* of the points P_3, Q_3; then on taking account of the fact that $(P_1\ Q_1,\ P_2\ Q_2,\ P_3\ Q_3)$ belong to an involution, this also implies that the harmonic conjugate of Q_2 with respect to $P_1\ Q_1$ is the *other* of the points P_3, Q_3. The envelope of the line will then be a contravariant of the three conics. To find its form, let $P_1\ Q_1$ be the roots of the binary quadratic $p_x^2 = p'^2_x$, $P_2\ Q_2$ of q_x^2 and therefore $P_3\ Q_3$ of $\lambda\ p_x^2 + \mu\ q_x^2$. The quadratic whose roots are the harmonic conjugates of P_2, Q_2 with respect to P_1Q_1 is easily seen to be

$$2(p\ q)^2 p_x^2 - (p\ p')^2\ q_x^2.$$

Since these harmonic conjugates are to be P_3, Q_3 we must have

$$2\mu\ (p\ q)^2 + \lambda\ (p\ p')^2 = 0.$$

Hence by using Clebsch's Principle the required ternary contravariant is

$$S'_4 = 2\mu\ (ab\ \omega)^2 + \lambda\ (aa'\ \omega)^2.$$

2. This contravariant S'_4 is symmetrical in the co-efficients of the two conics $S_2 = b_x^2$ and $S_3 = c_x^2$. Its first term $(ab\ \omega)^2$ is the ϕ—conic of S_1 or a_x^2 and S_2 and therefore touches the tangents to S_1 and S_2 at their common points; its second term is merely the conic S_1 in line co-ordinates. Hence S_4 the polar reciprocal of S'_4 with respect to S_1 must be of the form $\lambda'\ S_1 + \mu'\ S_2$. To actually evaluate S_4, we notice that the polar reciprocal of ω_α^2 with respect to a_x^2 is

$$a_x\ a'_x\ a_\alpha\ a'_\alpha\ {}^*.$$

Hence we have

$$S_4 = \{\ 2\mu\ (aba'')\,(aba''') + \lambda\ (aa'a'')\ (aa'a''')\ \}\ a''_x\ a'''_x.$$

Calling the first term of this expression F, we have

$$F = (aba'')\ (aba''')\ a''_x\ a'''_x$$

$$= (aba'')\ a'''_x\ \{\ (aba'')\ a'''_x - (aa'''\ a'')\ b_x + (ba'''\ a'')\ a_x\ \}$$

$$= (aba'')^2\ a'''^2_x - F - (aba'')\ (aa'''\ a'')\ a'''_x\ b_x.$$

* Page 291 : Grace and Young.

Hence using the usual notation for the invariants,

$$2F = A_{112}\, a_x^2 - (aba'')(aa'''a'')\, a_x'''\, b_x.$$

Calling the second term of this G, we have

$$G = (aba'')(aa'''a'')\, a_x'''\, b_x$$

$$= (aa'''a'')b_x\{(aba''')a_x'' - (aa''a''')b_x + (ba''a''')a_x\}$$

$$= (aa''a''')^2 b_x^2 - G - G.$$

Thus $\qquad G = \frac{1}{3} A_{111} b_x^2.$

And therefore $\qquad F = \frac{1}{2}(A_{112}\, a_x^2 - \frac{1}{3} A_{111}\, b_x^2).$

Now if we put a' for b in F, we obtain the second term in the expression for S_4. Thus the value of this second term is $\frac{1}{3} A_{111}\, a_x^2$. We have finally therefore

$$S_4 = \mu\,(A_{112}\, a_x^2 - \tfrac{1}{3} A_{111}\, b_x^2) + \frac{\lambda}{3} A_{111}\, a_x^2.$$

$$= \left(\mu A_{112} + \frac{\lambda}{3} A_{111}\right) a_x^2 - \frac{\mu}{3} A_{111}\, b_x^2.$$

3. We can shew that the three conics S_2, S_3, S_4 form a symmetrical triad with respect to S_1, that is, that S_2 would be derived from S_3, S_4 by the same process which derived S_4 from S_2, S_3. Let the conics of the four-point system be represented parametrically: t_1, t_2, t_3, t_4 being the parameters of S_1 S_2 S_3 S_4 and t_0 of that conic S_0 of the system, which is ex-harmonic to S_1, that is, which is such that the invariant A_{011} vanishes. Taking the special parametric representation in which p/q is the parameter of the conic $p\,a_x^2 + q\,b_x^2$, we have

$$t_1 = \infty,\ t_2 = 0,\ t_3 = \frac{\lambda}{\mu},$$

$$t_4 = -\frac{3\mu A_{112} + \lambda A_{111}}{\mu A_{111}},\quad t_0 = -\frac{A_{112}}{A_{111}}.$$

We see that the two cubics

$$0.x^3 + 0.x^2 + A_{111}x + A_{112},$$

$$\mu^2 A_{111}\, x^3 + 3\mu^2 A_{112}\, x^2 - \lambda x\,(3\mu A_{112} + \lambda A_{111}),$$

which differ only by constant multipliers from $(x - t_1)^2\ (x - t_0)$ and $(x - t_2)(x - t_3)\ (x - t_4)$ are identically apolar. Hence we have:—

The symmetrical triads S_2, S_3, S_4 are determined parametrically by cubics apolar to the fixed cubic $(x - t_1)^2\ (x - t_0)$, where t_1 is the parameter of the conic S_1, and t_0 of the ex-harmonic conic of S_1.

4. The geometrical meaning of the symmetrical triads $S_2\ S_3\ S_4$ is immediately obtained from the process which derived S_4. Thus, if A be any point on S_4 and if the polar of A with respect to S_1 cuts S_2, S_3 in BB′,CC′ respectively, then from the definition of S_4, BC and B′C′ are conjugate pairs with regard to S_1.

Hence ABC, AB′C′ are both self-polar triangles of S_1. Thus:

There exist an infinity of self-polar triangles of S_1 (two, for each vertex), with their vertices situated severally on S_2, S_3, S_4.

The conic S_0 which is circumscribed to an infinity of self-polar triangles of S_1 may be regarded as a repeated triad of the system of triads $S_2\ S_3\ S_4$; it arises in fact when the apolar cubic of $(x - t_1)^2 (x - t_0)$ is taken to be $(x - t_0)^3$.

5. In a paper entitled 'On Mr. Wilkinson's Rectangular Transformation' (Proc. L.M.S., Vol. 24, First Series), Cayley considered three cones of the form

$$(p + \lambda)x^2 + (q + \lambda)y^2 + (r + \lambda)z^2 = 0,$$
$$(p + \mu)x^2 + (q + \mu)y^2 + (r + \mu)z^2 = 0,$$
$$(p + \nu)x^2 + (q + \nu)y^2 + (r + \nu)z^2 = 0,$$

and shewed that they contain an infinity of rectangular axes, one in each if $p + q + r + \lambda + \mu + \nu = 0$.

This is substantially the same as the above results.

The Homographic break-up of certain derived Loci.*

BY A. NARASINGA RAO, M.A.

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Statement of the Problem considered.

1. Let C be a rational curve and $\phi(\lambda, \mu)$ a symmetric binary n-ic. If we select pairs of points on C whose parameters λ and μ are connected by the relation

$$\phi(\lambda, \mu) = 0, \quad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

and associate with such a pair a determinate space element (such as, the line joining them or the intersection of their osculating planes), we shall have an one-dimentional aggregate of these elements which we shall refer to as the *derived locus*.

Starting with any point λ the relation (1) gives us n *conjugate* points $(\mu_1\ \mu_2 \ldots \mu_n)$ with each of which it may be paired. Two points of the μ set (which have the common conjugate point λ) are called *associates*. Again for any point μ in this set, we obtain in the same manner, n conjugates $\lambda(\lambda_1\ \lambda_2 \ldots \lambda_n)$ of which the starting point must clearly be one. Two cases arise according as the point aggregates

$$\lambda,\ [\mu_1\ \ \mu_2\ \ldots\ldots\ \mu_n]$$
$$\mu,\ [\lambda_1\ \lambda_2\ \ldots\ldots\ \lambda_n]$$

are the same or different.

In the first case, the distinction between conjugates and associates breaks down, and each of a set of $n+1$ points is conjugate to all the rest while the others are, for the time being, associates. The parameters $x_1\ x_2\ x_3 \ldots\ldots x_{n+1}$ of such a set of points are the roots of an equation

$$f(x) + t\, g(x) = 0, \quad \ldots \qquad \ldots \quad (2)$$

the first member of which belongs to a definite pencil of $(n+1)$ ics.

If λ and μ be two roots of (2), we have on substitution in (2) and elimination of t

$$f(\lambda)\, g(\mu) - f(\mu)\, g(\lambda) = 0, \quad \ldots \qquad \ldots \quad (3)$$

---

* I am indebted to Mr. R. Vythynathaswami for valuable suggestions in connection with this investigation.

The first member of (3) is divisible by $\lambda - \mu$ and the other factor $\phi(\lambda, \mu)$ equated to zero gives the (n, n) relation connecting conjugate points.*

In the other case, we have distinct sets (λ_r) and (μ_r), the conjugates say of μ_1 and λ_1 respectively. The conjugate set of λ_2 (an associate of λ_1) consists of μ_1 and $(n-1)$ other points. Hence arise two cases according as the conjugate sets of λ_1 and λ_2 are the same or different. In this paper, we shall consider only the first case, that is, we assume that all associate points have the same conjugate set.† In other words, we take it that if two points have one conjugate in common they have $(n-1)$ others besides. The two pencils of n-ics of which the roots are (λ_r) and and (μ_r) respectively then become identical as the set (λ_r) passes over into the set (μ_r) whenever one member of the first set passes over into a member of the second.

Taking these to be the left-hand members of

$$f(x) + \alpha\, g(x) = 0, \quad \ldots \quad \ldots \quad (4)$$

and

$$f(x) + \beta\, g(x) = 0, \quad \ldots \quad \ldots \quad (5)$$

there is an involutric correspondance connecting α and β as each polynomial determines the other uniquely.‡ Taking λ and μ as roots of the equations (4) and (5) and

$$\alpha + \beta = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

as the involution, we have on eliminating α and β from (4), (5) and (6)

$$f(\lambda)\, g(\mu) + f(\mu)\, g(\lambda) = 0 \quad \ldots \quad \ldots \quad (7)$$

the relation connecting conjugate points for this case.

2. Neither of the relations (3), (7) can in general be replaced by simpler ones rational in both λ and μ; but when this is possible, the corresponding locus breaks up into algebraically distinct loci. In parti-

---

* A simple example of a case where this happens is the locus of the meets of tangents to a plane curve when the join of their points of contact passes through a fixed point.

† That is, that any three of the equalities

$$\phi(\lambda_1 \mu_1) = 0, \quad \phi(\lambda_1 \mu_2) = 0, \quad \phi(\lambda_2 \mu_1) = 0, \quad \phi(\lambda_2 \mu_2) = 0,$$

implies the fourth.

‡ The locus of the meets of perpendicular tangents of a plane curve is an example of this type. The sets (λ_r) and (μ_r) are the parameters of points whose tangents are parallel to the lines $y = \lambda z$ and $y = \mu z$, while the involution referred to is $\lambda\mu + 1 = 0$, the condition for perpendicularity.

cular, if $f(x)\,g(y)\pm f(y)\,g(x)$ can be resolved completely into n bilinear factors, we shall have an utter breakdown of the derived locus into the maximum number of components.

We shall now consider, when factorisation is possible, the number of components and the distribution of the derived elements on the several loci.

The Factorisation of f(x) g(y) — f(y) g(x).

3. Let $f(x)\,g(y)-f(y)\,g(x)$ be the product of n factors such as

$$A_r xy + B_r x + C_r y + D_r \quad \ldots \quad \ldots \quad (8)$$

so that the equality

$$\frac{f(x)}{g(x)} = \frac{f(y)}{g(y)} \quad \ldots \quad \ldots \quad (9)$$

holds in virtue of any one of the n relations

$$y = -\frac{B_r x + D_r}{A_r x + C_r}. \quad \ldots \quad \ldots \quad (10)$$

From (9) and (10), we infer that the rational function $f(x)/g(x)$ remains unaltered by all the sabstitutions of the group generated by the n homographies (10). But as $f(x)$ and $g(x)$ are of order n and the roots of $f(x)+t\,g(x)=0$ have to be carried into themselves by all the transformations of the group, its order cannot exceed n. Hence the transformations (10) form a group of order n.

Conversely if $f(x)/g(x)$ *is invariant for all the transformations of a finite homographic group of order n, the binary n-ic* $[f(x)\,g(y)-f(y)\,g(x)]$ *is resolvable into n lineo-linear factors. This condition is both necessary and sufficient.*

4. To determine the invariant functions of a group, we start with an arbitrary linear factor and subject it to all the transformations of the group. The product of all the transtorms is a polynomial

$$f(x) + tg(x)$$

where the parameter t is fixed uniquely by the given linear factor. The roots of the polynomial are merely permuted among themselves by each operation of the group and the polynomial is transtormed into a numerical multiple of itself, *viz.* $k[f(x)+tg(x)]$. Hence the ratio $f(x)/g(x)$ is an absolute invariant.

The break-up of the Generalised Polar Locus.

5. We are now in a position to discuss the degeneracy of an interesting geometrical locus, the generalisation of the polar of a point with

respect to a conic. We shall call the locus of the meets of tangents to a curve C at the extremities of a chord through a given point O the polar locus of O. Here the conjugates of a point P are those in which the line OP meets the curve and the distinction between *conjugates and associates* is only one of point of view. We find accordingly on taking O for origin that the relation between the parameters of conjugate points is

$$f(t)\, g(t') - f(t')\, g(t) = 0,$$

the co-ordinates of a point on the curve being

$$X = \frac{f(t)}{h(t)},\ Y = \frac{g(t)}{h(t)}. \qquad \ldots \qquad \ldots \qquad (11)$$

Any set of points collinear with O have their parameters given by $f(x) + tg(x) = 0$ and when these are carried into themselves by a group of homographic substitutions of finite order, we have linear relations connecting t and t_1 and corresponding component loci. There are thus 5 types of break-up corresponding to the cyclic and polyhedral groups.

6. Consider for example a rational curve of order n with two hyper-inflexional tangents of n-point contact. Taking these for axes and 0 and ∞ as the parameters of the points of contact, the equation of the curve is

$$X = \frac{t_n}{h(t)},\quad Y = \frac{1}{h(t)},$$

where $h(t)$ is a general n-ic in t.

The set of points given by $t^n + k = 0$ are all collinear with O and carried into themselves by the substitutions of the group

$$t' = w^r t \quad (w = e^{\frac{2l\pi}{n}},\ r = 0, 1, 2, \ldots n-1.) \qquad \ldots \qquad (12)$$

We shall denote the point of intersection of the tangents at t_1 and t_2 by the notation $(t_1 t_2)$. The nC_2 points $(w^r t, w^s t)$ arrange themselves on the following distinct loci:

		Locus	
(t, wt) $(t, w^{n-1} t)$ $\}$	n intersections	L_1	... (13)

[These two pts. trace the same locus since when t passes into $w^{n-1} t$ the configuration (t, wt) passes into $(w^{n-1} t, t)$; that is, the first point passes over into the second].

$(t, w^2 t)$ $(t, w^{n-2} t)$ $\}$	n intersections	L_2
...	... ...	...
...	... ...	...

If n be odd, we have $\frac{1}{2}(n-1)$ such loci corresponding to the n operations of the group other than identity. If n be even ($n = 2s$), we have $\frac{1}{2}(n-2)$ loci L_r, each passing through n of the intersections and a special locus

$(t, w^s t \text{ or } w^{n-s} t)$ ...... s intersections ......L_s

passing through only $\frac{1}{2}n$ points and geometrically of a different character from the other loci. The nC_2 intersections are thus accounted for in each case.

7. It is easy to construct curves the degeneracy of whose polar loci is connected in the same manner with the polyhedral groups. The procedure will be obvious from the study of the tetrahedral case which we now take up.

The twelve substitutions

$$t' = \pm t, \ \pm \frac{1}{t}, \ \pm i\frac{t+1}{t-1}, \pm i\frac{t-1}{t+1}, \ \pm \frac{t+i}{t-i}, \ \pm \frac{t-i}{t+i}, \ \dots (14)$$

leave the function

$$\frac{(t^4 - 2\sqrt{3}\, i\, t^2 + 1)^3}{(t^4 + 2\sqrt{3}\, i\, t^2 + 1)^3}$$

invariant.

Hence the pencil of collinear points for the 12-ic* given by

$$x = \frac{(t^4 - 2\sqrt{3}\, i\, t^2 + 1)^3}{h(t)}, \ y = \frac{(t^4 + 2\sqrt{3}\, i\, t^2 + 1)^3}{h(t)}, \qquad \dots (15)$$

where $h(t)$ is a polynomial of order 12 is carried into itselt by all the substitutions of (14).

The $\frac{12 \times 11}{2} = 66$ intersections of the tangents at the 12 points (14) arrange themselves on 7 loci as follows :

			Locus
$(t, -t)$	...	... 6 intersections	L_1
$\left(t, \frac{1}{t}\right)$	...	... 6 ,,	L_2
$\left(t, -\frac{1}{t}\right)$	...	... 6 ,,	L_3.

---

* The curve $x = \frac{f_n(t)}{h_n(t)}, y = \frac{g_n(t)}{h_n(t)}$ is not necessarily of order n as can be seen from the pseudo-quartic $x = t^2$, $y = t^4$ which consists of the parabola $y = x^2$ taken twice over. A reduction of degree of this sort can be avoided by a proper choice of $h(t)$.

A transformation H of period 3 carries t successively into

$$H(t) \text{ and } H^2(t)$$

and the three points

$$(t, H(t)) \quad (t, H^2 t) \quad (Ht, H^2 t)$$

all lie on the same locus since in each case the pair are connected by the relation $t' = H(t)$. The 12 points separate into 4 sets such as this, the members of each set being permuted among themselves by H. We have thus 4×3 or 12 points lying on the locus $t' = H(t)$ which is also the same as that given by $t' = H^2(t)$. Hence the following additional loci:—

$$\left\{\begin{matrix} t, & i\dfrac{t+1}{t-1} \\ t, & \dfrac{t+i}{t-i} \end{matrix}\right\} \quad \ldots \quad \ldots \quad 12 \text{ intersections } \quad L_4$$

$$\left\{\begin{matrix} t, & -i\dfrac{t+1}{t-1} \\ t, & \dfrac{t-i}{t+i} \end{matrix}\right\} \quad \ldots \quad \ldots \quad 12 \quad \text{"} \quad L_5$$

$$\left\{\begin{matrix} t, & \dfrac{i-t}{i+t} \\ t, & \dfrac{i(1-t)}{1+t} \end{matrix}\right\} \quad \ldots \quad \ldots \quad 12 \quad \text{"} \quad L_6$$

$$\left\{\begin{matrix} t, & \dfrac{i+t}{i-t} \\ t, & i\dfrac{t-1}{t+1} \end{matrix}\right\} \quad \ldots \quad \ldots \quad 12 \quad \text{"} \quad L_7.$$

Total number of intersections ... $(6\times3)+(12\times4) = 66.$

8. Working on the same lines we find that for *the dihedral break-up*, there are, as in the cyclic case, according as n is even or odd

$$\tfrac{1}{2}n \text{ or } \tfrac{1}{2}(n-1) \text{ loci}$$

corresponding to the single substitution of order n. Each of these loci passes however through twice as many points as before, thus accounting for $n(n-1)$ intersections. There are besides n other loci each passing through n points corresponding to the n involutions in the group.

For the *octahedral break-up,* we have

(i) 6 loci corresponding to the 6 involutions, each locus passing through 12 intersections ...	$12 \times 6 = 72$
(ii) 4 loci corresponding to the 4 homographies of order 3, each passing through 8 triads of intersections	$24 \times 4 = 96$
(iii) 3 loci on which lie the points (t, Ht) and (t, H^3t) corresponding to the 3 substitutions of period 4, each locus passing through 6 sets of 4 intersections	$24 \times 3 = 72$
and (iv) 3 loci on which lie (t, H^2t), each through 6 sets of 2 points	$12 \times 3 = 36$
Total ...	276

Altogether there are 16 loci on which the $\frac{24 \times 23}{2} = 276$ points of intersection of the tangents at a set of 24 points collinear with O distribute themselves.

Similarly we find that when the polar locus of a rational 60-ic breaks up *icosahedrally*, we have the following 37 partial loci:—

	No. of Intersections
(i) 15 loci corresponding to the 15 substitutions of period 2. On each of these lie 30 of the 1770 intersections determined by a line through O	450
(ii) 10 loci corresponding to the 10 substitutions of period 3, each passing through 20 sets of 3 intersections	$60 \times 10 = 600$
(iii) 6 loci on which lie the points (t, Ht) and (t, H^4t) H being any one of the 6 substitutions of period 5, each through 12 sets of 5 intersections	$60 \times 6 = 360$
(iv) 6 loci on which lie (t, H^2t) and (t, H^3t), each through 15 sets of 2 intersections ...	$60 \times 6 = 360$
Total ...	1770

9. It may be remarked before concluding this part of the subject that as the degeneracy of the derived locus is due to the existence of simpler birational relations between conjugate points, the nature of the new element or elements associated with each pair of points is a matter of indifference. Thus it is a type of break-up that we have been considering of which the polar locus furnishes but one instance. As other instances which readily suggest themselves, we may cite

(i) the locus of intersection of the circles of curvature at pairs of connected points;

(ii) the surface traced by the intersection of the osculating planes at points of a twisted curve whose join meets a given line.

We now pass on to loci of the second type the degeneracy of which depends on

The factorisation of f(x) g(y) + f(y) g(x).

10. Let $f(x)$ and $g(x)$ be two polynomials which are relatively prime. Any member of the pencil they determine may be thrown into the form

$$f(x) + \lambda g(x) \qquad \ldots \qquad \ldots \quad (16)$$

and will be referred to as the polynomial λ. (P_λ). An involution among the members of the pencil (16) associates with P_λ a certain P_μ. We may take the relation of λ to μ to be

$$\lambda + \mu = 0, \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

since we may without loss of generality suppose $f(x)$ and $g(x)$ in (16) to be the fixed members of the involution. We now enquire whether this result (the interchange of P_λ and P_μ) can be brought about by a linear transformation on x. If

$$y = -\frac{Bx + D}{Ax + C} \qquad \ldots \qquad \ldots \quad (18)$$

be one such, it carries the polynomial (16) into

$$f(y) - \lambda g(y). \qquad \ldots \qquad \ldots \quad (19)$$

On eliminating λ between (19) and (16), we have

$$f(x)\, g(y) + f(y)\, g(x) = 0$$

true, if $\qquad Axy + Bx + Cy + D = 0.$

Hence the necessary and sufficient condition for the factorisation to be possible is that there exist n linear transformations each of which carries $[f(x) + \lambda g(x)]$ into $[f(x) - \lambda g(x)]$ and vice-versa.

The Three Solutions.

11. Let us for a moment think of a polynomial as the aggregate of its n zeros. Marking these on the Argand plane and transferring them to the Neumann sphere by stereographic projection, we have on the sphere 2 sets of n points corresponding to the 2 polynomials P_λ and P_μ. The n homographic transformations referred to correspond to n rotations carrying the spherical surface into itself. Each of these rotations carries a point of one set into a point of the other and hence generate a group which consists of all rotations which carry the $2n$ points

$$\{ f(x) + \lambda\, g(x) \} \; \{ f(x) + \mu\, g(x) \}$$

into themselves. This group is clearly of order $2n$ and carries an arbitrary point on the sphere into 2 sets of n points each, n of the rotations carrying each set into itself and the other n interchanging the 2 sets.

A succession of rotations through an angle $\frac{1}{n}\pi$ about the same axis carries an arbitrary point into a configuration of $2n$ points on a small circle of the sphere. The odd and even vertices of this $2n$-gon constitute 2 sets which are interchanged by the n rotations through odd multiples of $\frac{1}{n}$ and carried into themselves by the n rotations through even multiples of $\frac{1}{n}\pi$ including one through 2π. The odd and even vertices can coalesce only at either of the 2 points where the axis cuts the sphere. These points each taken n times over are the double members of the pencil in involution. Hence *the $(n-n)$ relation determined by an involution between the polynomials of a pencil $f(x) + \lambda\, g(x)$ breaks up into the n homographies of a cyclic group when the fixed members of the pencil are perfect n^{th} powers and conversely.** ... ... (20)

12. We pass on next to the rotation group associated with the dihedron, the volumeless "solid" formed by a regular n-gon placed upon itself with vertices coincident. The dihedron is carried into itself by a group of $2n$ rotations — n through an angle $\left(\frac{2r\pi}{n}\right)$ about a line through the centre perpendicular to the plane of the polygon and n through angles π about the n lines of symmetry in the plane. Transferring points from the Neumann sphere to the 2 faces of the dihedron by orthogonal projection, we see that any point on the dihedron is carried over by these rotations into

* The converse amounts to a knowledge of the factorisation of $(x^n + y^n)$.

Notes and Questions.

Gupta's Perpetual Calendar for finding the Day of the Week.

KEY :—

We take the following steps :—

1. Divide the year in question by 400.

2. (*a*) Find out the letter at the head of the column in which the remainder lies. (See Table No. 1.)

   (*b*) If however the remainder be followed by a dot, and the month in question be not January or February, use the letter at the head of the column in which the dot lies.

3. Find this letter in the row of the month in question. (See Table No. 2.)

4. Look down the corresponding column and find the week day corresponding to the date in question. (See Table No. 3.)

EXAMPLES :—

Find the day of the week on :—

(1) 9th October 1902,

(2) 1st August 1920,

(3) 19th February 1840.

1. The remainders are 302, 320 and 240 respectively.

2 (*a*), (*b*) The corresponding letters are N, R, N respectively.

3. These occur in the 3rd., the last, and the 6th column respectively in T. 2.

4. Looking down the columns, we have :—

Thursday on 9-10-1902
Sunday on 1- 8-1920
Wednesday on 19- 1-1840.

TABLE NO. 1.

H	A	N	S	R.	J.	G.
1	2	3	4	.	5	6
7	8	.	9	10	11	12
.	13	14	15	16	.	17
18	19	20	.	21	22	23
24	.	25	26	27	28	.
29	30	31	32	.	33	34
35	36	.	37	38	39	40
.	41	42	43	44	.	45
46	47	48	.	49	50	51
52	.	53	54	55	56	.
57	58	59	60	.	61	62
63	64	.	65	66	67	68
.	69	70	71	72	.	73
74	75	76	.	77	78	79
80	.	81	82	83	84	.
85	86	87	88	.	89	90
91	92	.	93	94	95	96
.	97	98	99	100	101	102
103	104	.	105	106	107	108
.	109	110	111	112	.	113
114	115	116	.	117	118	119
120	.	121	122	123	124	.
125	126	127	128	.	129	130
131	132	.	133	134	135	136
.	137	138	139	140	.	141
142	143	144	.	145	146	147
148	.	149	150	151	152	.
153	154	155	156	.	157	158
159	160	.	161	162	163	164
.	165	166	167	168	.	169
170	171	172	.	173	174	175
176	.	177	178	179	180	.
181	182	183	184	.	185	186
187	188	.	189	190	191	192
.	193	194	195	196	.	197

198	199	200	201	202	203	204
.	205	206	207	208	.	209
210	211	212	.	213	214	215
216	.	217	218	219	220	.
221	222	223	224	.	225	226
227	228	.	229	230	231	232
.	233	234	235	236	.	237
238	239	240	.	241	242	243
244	.	245	246	247	248	.
249	250	251	252	.	253	254
255	256	.	257	258	259	260
.	261	262	263	264	.	265
266	267	268	.	269	270	271
272	.	273	274	275	276	.
277	278	279	280	.	281	282
283	284	.	285	286	287	288
.	289	290	291	292	.	293
294	295	296	.	297	298	299
300	301	302	303	304	.	305
306	307	308	.	309	310	311
312	.	313	314	315	316	.
317	318	319	320	.	321	322
323	324	.	325	326	327	328
.	329	330	331	332	.	333
334	335	336	.	337	338	339
340	.	341	342	343	344	.
345	346	347	348	.	349	350
351	352	.	353	354	355	356
.	357	358	359	360	.	361
362	363	364	.	365	366	367
368	.	369	370	371	372	.
373	374	375	376	.	377	378
379	380	.	381	382	383	384
.	385	386	387	388	.	389
390	391	392	.	393	394	395
396	.	397	398	399	400	.

TABLE NO. 2.

January ...	H	A	N	S	R.	J.	G.
February ...	R.	J.	G.	H	A	N	S
March ...	R.	J.	G.	H	A	N	S
April ...	A	N	S	R.	J.	G.	H
May ...	G.	H	A	N	S	R.	J.
June ...	S	R.	J.	G.	H	A	N
July ...	A	N	S	R.	J.	G.	H
August ...	J.	G.	H	A	N	S	R.
September ...	N	S	R.	J.	G.	H	A
October ...	H	A	N	S	R.	J.	G.
November ...	R.	J.	G.	H	A	N	S
December ...	N	S	R.	J.	G.	H	A

TABLE NO. 3.

Sun. ...	7, 14, 21, 28	6, 13, 20, 27	5, 12, 19, 26	4, 11, 18, 25	3, 10, 17, 24, 31	2, 9, 16, 23, 30	1, 8, 15, 22, 29
Mon. ...	1, 8, 15 22, 29	7, 14, 21, 28	6, 13, 20, 27	5, 12, 19, 26	4, 11, 18, 25	3, 10, 17, 24, 31	2, 9, 16, 23, 30
Tues. ...	2, 9, 16, 23, 30	1, 8, 15, 22, 29	7, 14, 21, 28,	6, 13, 20, 27	5, 12, 19, 26	4, 11, 18, 25	3, 10, 17, 24, 31
Wed. ...	3, 10, 17, 24, 31	2, 9, 16, 23, 30	1, 8, 15, 22, 29	7, 14, 21, 28	6, 13, 20, 27	5, 12, 19, 26	4, 11, 18, 25
Thurs....	4, 11, 18, 25	3, 10, 17, 24, 31	2, 9, 16, 23, 30	1, 8, 15, 22, 29	7, 14, 21, 28	6, 13, 20, 27	5, 12. 19, 26
Fri. ...	5, 12, 19, 26	4, 11, 18, 25	3, 10, 17, 24, 31	2, 9, 16, 23, 30	1, 8, 15, 22, 29	7, 14, 21, 28	6, 13, 20, 27
Satur. ...	6, 13, 20, 27	5, 12, 19, 26	4, 11, 18, 25	3, 10, 17, 24, 31	2, 9, 16, 23, 30	1, 8, 15 22, 29	7, 14, 21, 28

HANS R. GUPTA, B.A.

Self-Conjugate Triangles.

1. Consider the conic

$$\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0$$

circumscribed about the triangle of reference. If D, E, F be points on the sides of the triangle of reference ABC, the conditions that DEF may be self-conjugate with respect to the conic are

(i) AD, BE, CF should be concurrent,

(ii) the point of concourse should lie on the conic.

Expressed analytically these conditions take the following forms:—

(i) Let the equations of AD, BE, CF be

$$y = lz, \; z = mx, \; x = ny;$$

then these will be concurrent provided $lmn = 1$.

(ii) The point of concourse is

$$(1, \frac{1}{n}, m)$$

which will lie on the conic provided $u + vn + w/m = 0$.

2. Suppose $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$, $(x_3\, y_3\, z_3)$ are the vertices of a second inscribed triangle A′B′C′; and that the corresponding sides of ABC, A′B′C′ intersect at D, E, F. Then the conditions that DEF may be self-conjugate to the conic are found as under:—

The equation to B′C′ is written in the form

$$\frac{ux}{x_2\, x_3}+\frac{vy}{y_2\, y_3}+\frac{wz}{z_2\, z_3}=0 \qquad \ldots \qquad \ldots \quad (1)$$

seeing that this is satisfied by the co-ordinates of B′, C′. The equation of AD is thus obtained by putting $x = 0$ in the above and is

$$\frac{vy}{y_2\, y_3}+\frac{wz}{z_2\, z_3}=0. \qquad \ldots \qquad \ldots \quad (2)$$

Similarly BE and CF are

$$\frac{wz}{z_3\, z_1}+\frac{ux}{x_3\, x_1}=0, \qquad \ldots \qquad \ldots \quad (3)$$

$$\frac{ux}{x_1\, x_2}+\frac{vy}{y_1\, y_2}=0. \qquad \ldots \qquad \ldots \quad (4)$$

DEF will be self-conjugate, provided AD, BE, CF are concurrent on the conic.

Hence

$$x_1x_2y_2y_3z_3z_1 + x_3x_1y_1y_2z_2z_3 = 0$$

i.e.,
$$x_2\, y_3\, z_1 + x_3\, y_1\, z_2 = 0. \quad \ldots \quad \ldots \quad (5)$$

The point of concourse may be obtained from (2), (3), (4) in a symmetrical form by observing

$$\left.\begin{array}{llll} \text{(i)} & ux = -x_1\, x_2\, x_3, & vy = y_1\, y_2\, x_3, & wz = x_2 z_1 z_3\,; \\ \text{(ii)} & ux = x_1 x_2\, y_3, & vy = -y_1\, y_2\, y_3, & wz = y_1\, z_2\, z_3\,; \\ \text{(iii)} & ux = z_2\, x_1\, x_3, & vy = z_1\, y_2\, y_3, & wz = -z_1\, z_2\, z_3\,; \end{array}\right\}$$

so that

$$\left.\begin{array}{l} ux = -\dfrac{(x_1\, x_2\, y_3)(x_1\, x_3\, z_2)}{x_1\, x_2\, x_3} = -(x_1\, y_3\, z_2), \\ vy = -(y_2\, z_1\, x_3), \\ wz = -(z_3\, x_2\, y_1). \end{array}\right\} \ldots \quad (6)$$

3. Conversely, if condition (5) holds, it follows that an inscribed triangle A′B′C′ will be such that its corresponding sides pass through D, E, F. Hence we have the theorem that *an infinite number of triangles can be inscribed in the conic having the corresponding sides passing through D, E, F—the vertices of a self-conjugate triangle.*

[This may be seen *directly* by writing the conic with respect to which the given triangle is self-conjugate as $ux^2 + vy^2 + wz^2 = 0$; so that, if $P\,(x_1\, y_1\, z_1)$ is a point on it then $Q\,(x_1, -y_1, z_1)$ and $R\,(x_1, y_1, -z_1)$ are also points on the same, with the obvious properties that the sides of P Q R pass through A, B, C respectively.]

4. Next, consider the conic

$$S \equiv ux^2 + vy^2 + wz^2 = 0, \quad \ldots \quad \ldots \quad (7)$$

and two triangles, ABC, A′B′C′ self-conjugate with respect to the same. Let A′B′C′ be $(x_1\, y_1\, z_1)$, $(x_2\, y_2\, z_2)$, $(x_3\, y_3\, z_3)$ referred to ABC as the triangle of reference; and let BC, B′C′ meet at D and so on. Then, we know from § 2 that DEF will be self-conjugate for the conic through ABCA′B′C′, if $x_2\, y_3\, z_1 + x_3\, y_1\, z_2 = 0$.

Now, the equation to EF is readily seen to be

$$ux\, x_2\, x_3 + vy\, x_2\, y_3 + wz\, x_3\, z_2 = 0. \quad \ldots \quad (8)$$

Also, if AA′, BB′, CC′ form the triangle PQR,

$$P \equiv \text{intersection of BB}' \ \&\ \text{CC}'$$

$$\equiv \quad " \quad \left[\frac{x}{x_2} = \frac{z}{z_2},\ \frac{x}{x_3} = \frac{y}{y_3}\right]$$

$$\therefore \quad P \equiv (x_2\, x_3,\ x_2\, y_3,\ x_3\, z_2), \quad \ldots \quad \ldots \quad (9)$$

which is identical with the pole of (8) with respect to (7).

But, from the harmonic property of DEF relative to the quadrangle BB′ CC′, it follows that the point P lies on EF. Hence, the straight line EF touches the conic (7) at the point P.

Similarly, FD, DE touch (7) at Q, R respectively.

5. Thus we have the following property :—

Let σ be the conic through the vertices of two triangles Δ_1, Δ_2 self conjugate with respect to a conic S; and let Δ_3 be the triangle formed by the intersections of corresponding sides of Δ_1, Δ_2; and Δ_4, that formed by joins of corresponding vertices of the same. Then, if Δ_3 is self-conjugate with respect to σ, then Δ_3, Δ_4 are in perspective and are circumscribed to and inscribed in S.

6. Writing the equation (7) in the form

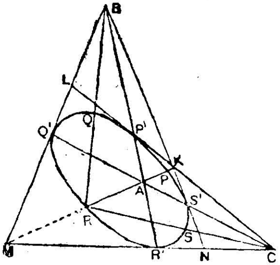
$$\frac{2x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0, \quad \ldots \quad \ldots \quad (10)$$

we find that the set of points $[p, \pm q, \pm r]$ has an interesting property, *viz.* the diagonal triangle of the quadrilateral formed by them is self-conjugate with respect to (10). For, the two diagonals given by

$$\frac{y^2}{q^2} = \frac{z^2}{r^2}, \quad \ldots \quad \ldots \quad (11)$$

will determine with the third diagonal $x = 0$ a self-conjugate triangle, provided equation (11) denotes two conjugate lines. This is obviously the case, since the pole of $\left(\frac{y}{q} = \frac{z}{r}\right)$ lies on $\left(\frac{y}{q} + \frac{z}{r}\right) = 0$.

7. Hence, given a conic and a self-conjugate triangle ABC, there exist two real sets of four points (PQRS), (P′Q′R′S′), such that (i) the harmonic triangle of PQRS is ABC and the diagonal triangle ATT′ of the quadrilateral PQRS is self-conjugate for the conic; (ii) the diagonal triangle of P′Q′R′S′ is ABC and the harmonic triangle of the same is ATT′.



The figure explains the mutual relations of the two sets.

HEMRAJ.

On the Mean Value Theorem.

1. If $f(x)$ is continuous for all values of x in the interval

$$a \leqslant x \leqslant b,$$

and has derivatives of the first n orders throughout the interval, the nth mean value theorem for $f(x)$ may be deduced by the successive application of Rolle's Theorem to the function $F(x)$ and its first $(n-1)$ derivatives, where

$$F(x) \equiv \begin{vmatrix} f(b) & \frac{b^n}{n!} & \frac{b^{n-1}}{(n-1)!} & . & , & b & 1 \\ f(a) & \frac{a^n}{n!} & \frac{a^{n-1}}{(n-1)!} & . & . & a & 1 \\ f'(a) & \frac{a^{n-1}}{(n-1)!} & \frac{a^{n-2}}{(n-2)!} & . & . & 1 & 0 \\ f''(a) & \frac{a^{n-2}}{(n-2)!} & \frac{a^{n-3}}{(n-3)!} & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ f^{(n-1)}(a) & a & 1 & . & . & 0 & 0 \\ f(x) & \frac{x^n}{n!} & \frac{x^{n-1}}{(n-1)!} & . & . & x & 1 \end{vmatrix}$$

since $F(a) = 0 = F(b)$ obviously.

2. If, however, a *single* application only of Rolle's Theorem is desired, consider the function

$$F(x) \equiv \left[\frac{(b-a)^n}{n!}\phi(x) - \frac{(b-x)^n}{n!}\phi(a)\right],$$

where

$$\phi(x) \equiv \begin{vmatrix} f(b), & \frac{b^{n-1}}{(n-1)!} & . & . & b, & 1 \\ f(x), & \frac{x^{n-1}}{(n-1)!} & . & . & x, & 1 \\ f'(x), & \frac{x^{n-2}}{(n-2)!} & . & . & 1, & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ f^{(n-1)}(x) & 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

so that $\phi(b) = 0$, $F(a) = 0$, $F(b) = 0$.

3. The evaluation of the determinants in the final result is rendered easy by the application of the identity

$$b^r \equiv a^r + r.(b-a)a^{r-1} + \frac{r(r-1)}{2!}.(b-a)^2 a^{r-2} + \ldots + (b-a)^r,$$

and the details are left to the reader.

F. H. V. GULASEKHARAM.

Solutions.

Question 833.

(M. K. KEWALRAMANI, M.A.):—If p and q be odd positive integers and $q \geqslant p + 1$, prove that the area of the loop in the first quadrant of the curve $x^q + y^q = 3ax^{q-p-1}\,y^p$ is

$$\frac{9a^2}{2q}\,\Gamma\left(\frac{2p+1}{q}\right)\,\Gamma\left(\frac{2(q-p)-1}{q}\right).$$

Solution by S. Audinarayanan.

Put $\qquad x = r\cos\theta,\ y = r\sin\theta.$

The equation gets transformed into

$$r(\cos^q\theta + \sin^q\theta) = 3a\cos^{q-p+1}\theta\,\sin^p\theta.$$

For the loop in the first quadrant, the limits of θ are 0 and $\frac{1}{2}\pi$.

$$\therefore\ \text{Area of the loop} = \tfrac{1}{2}\int^{\frac{1}{2}\pi} r^2\,d\theta$$

$$= \frac{9a^2}{2}\int \frac{\cos^{2q-2p-2}\theta\,\sin^{2p}\theta\,d\theta}{(\cos^q\theta+\sin^q\theta)^2}$$

$$= \frac{9a^2}{2}\int \frac{\tan^{2p}\theta\,.\,\sec^2\theta\,d\theta}{(1+\tan^q\theta)^2}$$

$$= \frac{9a^2}{2q}\int_0^\infty \frac{y^{\frac{2p-q+1}{q}}\,dy}{(1+y)^2},\quad \text{where } y = \tan^q\theta$$

$$= \frac{9a^2}{2q}\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \frac{9a^2}{2q}\,\Gamma(m)\,\Gamma(n)/\Gamma(m+n),$$

where $m = \dfrac{2p+1}{q}$ and $n = \dfrac{2(q-p)-1}{q}$.

Question 1079.

(A. C. L. WILKINSON):—Let ABCD be any four co-planar points (non-cyclic). Let α be the isogonal conjugate of A with respect to the triangle BCD, etc. Denote the common pedal circle of A, α with respect

to BCD by (A); and let O be the common point of intersection of (A), (B), (C), (D). Prove that :—

(1) the six joins of the four points A, B, C, D bisect at right angles the six joins of the four points α, β, γ, δ, *viz.*, AB bisects $\gamma\delta$ at right-angles, and that the meet of AB, $\gamma\delta$ is the point of intersection of the circles (C), (D) other than O ;

(2) the seven points of intersection of the four circles (A), (B), (C), (D) lie on a conic whose axes are parallel to the asymptotes of the rectangular hyperbola through A, B, C, D and which also passes through the centre of the rectangular hyperbola through α, β, γ, δ ;

(3) this conic will be a rectangular hyperbola if two of the sides of the quadrilateral ABCD are parallel ;

(4) the sum of the reciprocals of the powers of A, B, C, D with respect to (A), (B), (C), (D) respectively is zero.

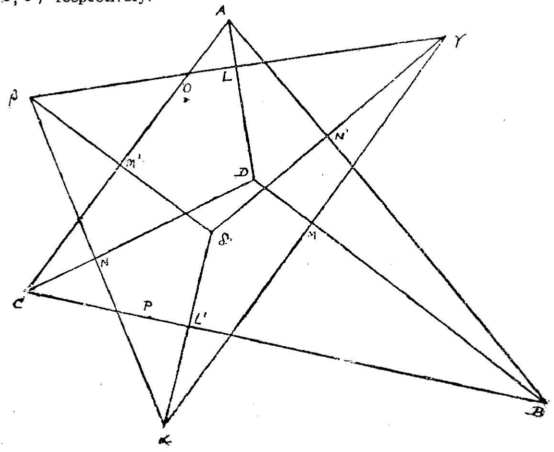
Solution by F. H. V. Gulasekharam.

(1) Now $\gamma\hat{B}A = C\hat{B}D = \delta\hat{B}A$

and $\gamma\hat{A}B = C\hat{A}D = \delta\hat{A}B$.

$\therefore$ AB bisects $\gamma\delta$ at right angles, and so on.

Let L, M, N, L′, M′, N′ be the middle points of $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, $\delta\alpha$, $\delta\beta$, $\delta\gamma$ respectively.



Then the circle (C) passes through L, M, N′; and the circle (D) passes through L′ M′ N′; so that N′, the meet of AB, $\gamma\delta$ is the point of intersection of the circles (C), (D) other than O.

(2) It is well-known that the six points L, M, N, L′, M′, N′—the six points of intersection other than O of the four circles (A), (B), (C), (D)—lie on a conic; *viz.* the centre-locus of all conics through α, β, γ, δ, which obviously passes through the centre of the R. H. $\alpha\beta\gamma\delta$ and has its axes parallel to the asymptotes of the R. H. (Smith's *Conic Sections*, Art. 213). We can easily verify that the conic passes through the common point O.

For since the circles L′MN, LM′N, LMN′ and L′M′N′ meet at O and since $M'\hat{L}N = M\hat{L'}N'$ and $M\hat{L}N' = N\hat{L'}M'$, it is readily seen that

$$O\,(M'\,N\,M\,N') = L'\,(M'\,N\,M\,N');$$

so that the points O, L′, M, M′, N, N′ lie on a conic. The axes of this conic are also parallel to the asymptotes of the R. H. ABCD; for the rectangular hyperbolas ABCD and $\alpha\beta\gamma\delta$ are similar and similarly situated.*

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* Take $xy = c^2$ for the equation to the R. H. $\alpha\beta\gamma\delta$.

Let α, β, γ, δ be the points $\left(ct_1, \frac{c}{t_1}\right)$, etc.

Then since A is the circumcentre of the triangle $\beta\gamma\delta$, the co-ordinates of A are given by

$$\frac{c}{2}\left[\, t_2 + t_3 + t_4 + \frac{1}{t_2 t_3 t_4}\right], \frac{c}{2}\left[\frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + t_2\, t_3\, t_4\,\right];$$

similarly for B, C and D.

Hence the R. H. ABCD is given by the equation

$$\left[x - \frac{c}{2}(t_1 + t_2 + t_3 + t_4)\,\right]\left[y - \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\,\right]$$

$$+ \frac{c^2}{4\,t_1\,t_2\,t_3\,t_4}\,\left[1 - t_1\,t_2\,t_3\,t_4\,\right]^2 = 0.$$

So that the rectangular hyperbolas ABCD and $\alpha\beta\gamma\delta$ are similar and similarly situated.

Note:—The common point O of the four pedal circles is the centre of the R. H. ABCD; *i.e.* the pt. whose co-ordinates are

$$\frac{c}{2}(t_1 + t_2 + t_3 + t_4)\,,\ \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right).$$

N. B.—The equation of the conic LMNL′M′N′ which is the Nine points conic of $\alpha\beta\gamma\delta$ is

$$x\,(x - \frac{c}{2}\,\Sigma\, t_1) = t_1\, t_2\, t_3\, t_4\, y\,(y - \frac{c}{2}\,\Sigma\, t_1^{-1})$$

which evidently passes through $\left(\frac{c}{2}\,\Sigma\, t_1\,, \frac{c}{2}\,\Sigma\, t_1^{-1}\right)$, the centre of the rectangular hyperbola ABCD.—(Ed.)

(3) The conic OLL′MM′NN′ will be *a pair of straight lines* if two of the sides of the quadrilateral $\alpha\beta\gamma\delta$ and consequently if two of the sides of the quadrilateral ABCD are parallel (Smith's *Conic Sections*, Art. 212). There appears to be an error in the question as stated.

(4) Let p_1 be the power of A with respect to the circle (A), and let p_2, p_3, p_4 have similar meanings.

Let r_1, r_2 be the lengths of the semi-major and semi-minor axes of the inconic of the triangle BCD, having A, α for foci.

Then $p_1 = (r_1^2 - r_2^2) - r_1^2 = -r_2^2 = \text{AP} \cdot \alpha\text{L}'$ (where AP is the perpendicular from A on BC).

$$\therefore \quad p_1 = -\text{AP} \cdot \delta\text{L}' = -\delta\text{L}' \cdot \frac{2\,\Delta\,\text{ABC}}{\text{BC}}$$

$$\therefore \quad \frac{\text{BC}}{\delta\text{L}'} = -\frac{2\,\Delta\,\text{ABC}}{p_1};$$

similarly $$\frac{\text{CA}}{\delta\text{M}'} = -\frac{2\,\Delta\,\text{ABC}}{p_2},$$

and $$\frac{\text{AB}}{\delta\text{N}'} = -\frac{2\,\Delta\,\text{ABC}}{p_3}.$$

Again, since the perps. from D on BC, CA, AB are of lengths

$$\frac{p_4}{\delta\text{L}'}, \frac{p_4}{\delta\text{M}'}, \frac{p_4}{\delta\text{N}'} \text{ respectively.}$$

$$\therefore \quad \frac{\text{BC}}{\delta\text{L}'} + \frac{\text{CA}}{\delta\text{M}'} + \frac{\text{AB}}{\delta\text{N}'} = \frac{2\,\Delta\,\text{ABC}}{p_4}.$$

$$\therefore \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 0.$$

Question 1107.

(CHARLES SALDANHA):—If n be any odd integer not divisible by 3, the integral part of $\{4^n - (2+\sqrt{2})^n\}$ is a multiple of 112.

Remarks by N. Maitra.

The proof given of this theorem on page 236 of Vol. XIV of the Journal is not a rigid one. There it is first proved that the expression is divisible by 8, and then that it is divisible by 14, and the conclusion is drawn that it is, therefote, divisible by 112. The only legitimate conclusion is, however, that it is divisible by 56.

Let the rational part in the expansion of $(2+\sqrt{2})^n$ be denoted by α and the co-efficient of the irrational part by β. Then,

$$(2+\sqrt{2})^n = \alpha + \beta\sqrt{2}, \ (2-\sqrt{2})^n = \alpha - \beta\sqrt{2},$$

$$\{4^n - (2+\sqrt{2})^n - (2-\sqrt{2})^n\} = 4^n - 2\alpha$$

$$= 2\,(2^{2n-1} - \alpha).$$

But $\alpha = \frac{n(n-1)}{1 \cdot 2} \cdot 2^{n-2} \cdot (\sqrt{2})^3 + \ldots\ldots + n \cdot 2 \cdot (\sqrt{2})^{n-1}$.

Here, the powers of 2 are in the descending order and the lowest power occurs in the last term. The exponent of 2 in the last term is $(n+1)/2$, so that for values 5 and over, of n, the last term is divisible by 8 and consequently, all other terms and therefore α also, are divisible by 8. 2^{2n-1} is also, clearly, divisible by 8. Therefore, for values 5 or more, of n, $2(2^{2n-1} - \alpha)$ is divisible by 16.

The only odd number less than 5 which is not divisible by 3 is 1. If $n = 1$,

$$\{ 4^n - (2+\sqrt{2})^n - (2-\sqrt{2})^n \} = 0.$$

Hence if n be any odd integer not divisible by 3, the given expression is divisible by 16.

As proved on pages 236—7 of the Journal, the expression is divisible by 14 under the conditions stated. Therefore, it is also divisible by 112.

Question 1134.

(N. B. MITRA) :—If p is a positive integer not equal to 0 (mod. 3), prove that

$$\sum_{1}^{p} \{ C_{2r} \, 2^{p-r} (2^{r} - 1) \} \equiv 0 \ (\text{mod. } 7),$$

where C_r denotes ${}_{2p+1}C_r$.

Solution by K. Satyanarayana.

We have

$$\sum_{r=1}^{p} C_{2r} \, 2^{p-r} (2^{r} - 1)$$

$$= 2^{p} \sum_{1}^{p} C_{2r} - \frac{1}{\sqrt{2}} \sum_{1}^{p} C_{2r} \, (\sqrt{2})^{2p+1-2r}$$

$$= 2^{p} (2^{2p} - 1) - \frac{1}{\sqrt{2}} \left\{ \tfrac{1}{2} \left[(\sqrt{2}+1)^{2p+1} + (\sqrt{2}-1)^{2p+1} \right] - (\sqrt{2})^{2p+1} \right\}$$

$$= 2^{3p} - \frac{1}{2\sqrt{2}} \left[(\sqrt{2}+1)^{2p+1} + (\sqrt{2}-1)^{2p+1} \right]$$

$$= \frac{(2\sqrt{2})^{2p+1} + (-\sqrt{2}-1)^{2p+1} + (-\sqrt{2}+1)^{2p+1}}{2\sqrt{2}}$$

The problem is to show that the numerator is a multiple of 7.

Applying Cauchy's Theorem for $(a^n + b^n + c^n)$ in terms of $\Sigma\, ab$ and abc when $a + b + c = 0$, we find that numerator is a multiple of $\Sigma\, ab$ or 7, only when p is of the form $3n - 1$. Hence p should not be of the form $3n + 1$.

Question 1198.

(F. H. V. GULASEKHARAM) :—The major axis of an inscribed conic of a triangle ABC is given by the trilinear equation

$$p\alpha + q\beta + r\gamma = 0.$$

(i) Prove that the equation of the conic is

$$\Sigma \{ p\,(p^2 - q^2 - r^2 + 2qr \cos A)\, \alpha \}^{\frac{1}{2}} = 0.$$

(ii) If the axis be fixed in direction, the centre traces out a rectangular hyperbola (through the in-centre and the ex-centre), *w.r.t.* which the original triangle is self-polar.

Solution by the Proposer and V. A. Mahalingam, B.A. (Hon.)

(i) Let (f, g, h) be a focus.

Then the tangential equation to the in-conic is

$$(lf + mg + nh)\,(l/f + m/g + n/h)$$
$$= l^2 + m^2 + n^2 - 2\,mn \cos A - 2\,nl \cos B - 2\,lm \cos C;$$

which reduces to the form

$$\Sigma \left[\frac{g/h + h/g + 2 \cos A}{l}\right] = 0.$$

Now remembering that

$$\left.\begin{aligned} pf + qg + rh &= 0 \\ p/f + q/g + r/h &= 0. \end{aligned}\right\}$$

and

we get $$g/h + h/g = \frac{p^2 - q^2 - r^2}{qr}\text{; etc.}$$

Hence the above tangential equation becomes

$$\Sigma \left[\frac{p\,(p^2 - q^2 - r^2 + 2qr \cos A}{l}\right] = 0,$$

whence the pt. equation is immediately written down.

To prove (ii) : let the major axis be always at right angles to the fixed straight line $l\alpha + m\beta + n\gamma = 0$; and let α, β, γ be the centre. Then if

$$L \equiv l - m \cos C - n \cos B$$
$$M \equiv m - n \cos A - l \cos C$$
$$N \equiv n - l \cos B - m \cos A,$$

the foci are given by

$$(\alpha \pm k\,L,\ \beta \pm k\,M,\ \gamma \pm k\,N),$$

where $\alpha^2 - k^2 L^2 = \beta^2 - k^2 M^2 = \gamma^2 - k^2 N^2$

$$= (\text{semi-minor-axis})^2.$$

Eliminating k^2 between the last set of equations, we get

$$\alpha^2 (M^2 - N^2) + \beta^2 (N^2 - L^2) + \gamma^2 (L^2 - M^2) = 0,$$

which is the equation to the locus of the centre. Obviously it is a R. H. self-polar to the triangle of reference. It also passes through the points $(\pm 1, \pm 1, \pm 1)$.

Question 1200.

(MARTYN M. THOMAS):—Show that the parabola

$$(a^2 y^2 - b^2 x^2) \sin 2\theta - 2\,abxy \cos 2\theta + b^2 x^2 + a^2 y^2 - 4\,a^2 b^2$$
$$+ 2\sqrt{2}.\,ab\,\{ (bx - ay) \sin \theta + (bx + ay) \cos \theta \} = 0,$$

where θ is variable, osculates its envelope.

Solution by the Proposer.

Changing θ to $\frac{\pi}{4} - \alpha$ and dividing throughout by $a^2 b^2$, the given equation becomes

$$\left(\frac{y^2}{b^2} - \frac{x^2}{a^2}\right) \cos 2\alpha - 2\,\frac{xy}{ab} \sin 2\alpha + \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$
$$- 4 + \frac{2\sqrt{2}}{ab}\left\{ (bx - ay).\,\frac{1}{\sqrt{2}}\,(\cos\alpha - \sin\alpha) \right.$$
$$\left. + (bx + ay)\,\frac{1}{\sqrt{2}}\,(\cos\alpha + \sin\alpha) \right\} = 0.$$

Removing the common factor 2, we have

$$\frac{x^2}{a^2} \sin^2\alpha - \frac{2xy}{ab} \sin\alpha \cos\alpha + \frac{y^2}{b^2} \cos^2\alpha + \frac{2x}{a} \cos\alpha + \frac{2y}{b} \sin\alpha - 2 = 0,$$

which clearly represents a parabola, since the terms of the second degree form a perfect square.

This equation may be thrown into the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 - \left(\frac{x}{a} \cos\alpha + \frac{y}{b} \sin\alpha - 1\right)^2 = 0. \qquad \ldots (1)$$

Since $\frac{x}{a} \cos\alpha + \frac{y}{b} \sin\alpha - 1 = 0$ is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

for different values of α, it follows that the parabola (1) has contact of the third order with the ellipse at the point whose eccentric angle is α.

Hence the given equation represents a variable parabola osculating its envelope which is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Question 1204.

(A. A. KRISHNASAWMI IYENGAR):—Conics are drawn having quadritactic contact with the Folium of Descartes at one point and bitactic contact elsewhere. Show that the chord of contact envelopes a Rectangular Hyperbola.

Solution by Martyn M. Thomas, M.A. and P. R. Venkatakrishna Iyer.

Equation to the Folium of Decartes is $x^3 + y^3 = kxy$.

Its parametric equation (obtained by putting $y = \lambda x$) is

$$x = \frac{k\lambda}{1+\lambda^3},\ y = \frac{k\lambda^2}{1+\lambda^3}. \quad \ldots \quad \ldots (1)$$

Equation to any conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \ \ldots \quad \ldots (2)$$

Wherever this conic meets the curve, (1) must satisfy (2). Substituting for x and y from (1) in (2), and arranging

$$c\lambda^6 + 2fk\lambda^5 + (bk^2 + 2gk)\lambda^4 + (2hk^2 + 2c)\lambda^3$$
$$+ (ak^2 + 2fk)\lambda^2 + 2gk\lambda + c = 0,$$

which is a sextic in λ, giving six values of λ, showing thereby the six possible points of intersection of the Conic and the Folium.

If the conic has quadritactic contact at P and bitactic contact at Q, there will be four coincident values at P and two other coincident values at Q.

Thus if P be λ, and Q be μ, the product of the six roots of the sextic equation, namely, $\lambda^4\mu^2 = 1$. $\therefore \quad \mu = \frac{1}{\lambda^2}$.

The point Q is given by

$$x = \frac{k\mu}{1+\mu^3} = \frac{k\lambda^4}{1+\lambda^6} \text{ and } y = \frac{k\mu^2}{1+\mu^3} = \frac{k\lambda^2}{1+\lambda^6}. \quad \ldots (4)$$

Equation to the chord of contact PQ is $\begin{vmatrix} x, & y, & 1 \\ k\lambda, & k\lambda^2, & 1+\lambda^3 \\ k\lambda^4, & k\lambda^2, & 1+\lambda^6 \end{vmatrix} = 0,$

which gives, $x\lambda^4 - k\lambda^2 + y = 0$, after dividing by the factor $k\lambda\,(\lambda^3 - 1)$.

Hence the envelope of PQ is the Rectangular Hyperbola $4xy = k^2$.

Question 1212.

(A. NARASINGA RAO) :—Show that if the pedal lines of PQR with respect to the triangle ABC are concurrent, the pedal lines of ABC with respect to the triangle PQR are also concurrent.

Geometrical Solution by B. Venkatesvaralu and Analytical Solution by K. Satyanarayana.

[The question is identical with Q. 178 of Mr. S. Narayanan solved on p. 81 of Vol. II, J.I.M.S.—Ed.]

Question 1222.

(A. A. KRISHNASWAMI AIYANGAR) :—If $a_1(1+a_2) = a_2(1+a_3) = \dots\dots = a_{n-1}(1+a_n) = a_n(1+a_1)$, prove that

$$\text{each} = -\tfrac{1}{4}\sec^2\frac{r\pi}{n},$$

where r may take any value from 1 to $\frac{1}{2}(n-2)$ when n is even or any value from 1 to $\frac{1}{2}(n-1)$ when n is odd.

Solution by K. Satyanarayana.

Let $a_1(1+a_2) = a_2(1+a_3) = \dots = a_{n-1}(1+a_n) = a_n(1+a_1) = k$. Then

$$a_2 = -1 + \frac{k}{a_1};\; a_3 = -1 + \frac{k}{-1+}\,\frac{k}{a_1}, \dots\dots;$$

$$a_n = -1 + \frac{k}{-1+}\,\frac{k}{-1+}\dots+\frac{k}{-1}+\frac{k}{a_1},$$

the k's being $(n-1)$ in number.

$$\therefore \qquad a_1 = -1 + \frac{k}{-1+}\,\frac{k}{-1+}\dots+\frac{k}{-1+}\,\frac{k}{a_1}; \qquad (1)$$

the k's being n in number.

But in the C.F. $\left[-1+\frac{k}{-1+}\,\frac{k}{-1+}\dots\right]$.

$$p_n + p_{n-1} - kp_{n-2} = 0,\; q_n + q_{n-1} - kq_{n-2} = 0.$$

$\therefore$ p_n, q_n are respectively the co-efficients of x^n in

$$\frac{1}{1+x-kx^2} \text{ and } \frac{x}{1+x-kx^2}.$$

$$\therefore \qquad p_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta};\; q_n = \frac{\alpha^n-\beta^n}{\alpha-\beta},$$

where $\alpha+\beta = -1$, $+\alpha\beta = -k$.

$\therefore$ by (1) $$a_1 = \frac{a_1(\alpha^{n+1} - \beta^{n+1}) + k(\alpha^n - \beta^n)}{a_1(\alpha^n - \beta^n) + k(\alpha^{n-1} - \beta^{n-1})}$$

i.e., $$(k - a_1^2 - a_1)(\alpha^n - \beta^n) = 0,$$

Now k is not equal to $a_1^2 + a_1$, as a_1, a_2, a_3, etc. are different quantities.

$$\therefore \qquad \alpha^n = \beta^n, \text{ or } \left(\frac{\alpha}{-1-\alpha}\right)^n = 1.$$

$$\therefore \qquad \frac{\alpha}{-1-\alpha} = \left(\cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}\right),$$

r having any value from 0 to $(n-1)$.

Hence $$k = -\alpha\beta = \alpha(1+\alpha) = -\tfrac{1}{4}\sec^2\frac{r\pi}{n},$$

on simplification.

Question 1224.

(A. A. KRISHNASWAMI AIYANGAR) :—If H be the in-Feurbach point and X, Y, Z, the middle points of the sides AB, BC, CA of the triangle ABC, prove that, with a suitable modification of signs,

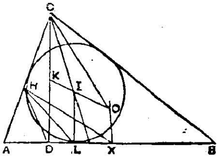
$$HX + HY + HZ = 0.$$

Solution by M. V. Ramakrishnan.

Let O be the circum-centre and I the in-centre.

Let CD be drawn perpendicular to AB, and let OI meet CD in K.

Now, since the in-circle touches DX at L, HL bisects the angle XHD.



$\therefore$ HX : HD = XL : LD
= OI : IK
= OC : CK.

Also $\angle$XHD = A — B = OCK.

$\therefore$ Δs OCK, XHD are similar.

$\therefore$ HX : XL = OC : OI

But XL = $\frac{1}{2}(a-b)$.

$\therefore$ HX varies as $(a-b)$

= $k(a-b)$, say,

Similarly HY = $k(b-c)$ and HZ = $k(c-a)$.

Hence the result.

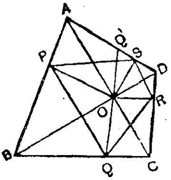
Question 1225.

(B. B. BAGI) :—ABCD is a quadrilateral in which the diagonals BD, AC intersect in O at right angles. OP, OQ, OR, OS are drawn perpendicular to AB, BC, CD, DA meeting them in P, Q, R, S and the sides CD, DA, AB, BC in P′, Q′, R′, S′. Show that PQRS, P′Q′R′S′ are concyclic.

Solution by P. R. Venkatakrishna Iyer, V. A. Apte, K. Satyanarayana, S. Audinarayanan and K. Srinivasa Raghavan.

If the angles of the quadrilateral ABCD are denoted by A, B, C, D, then

$$\angle OPQ + \angle OPS = \angle OBQ + \angle OAD = A+B-90^\circ \quad \ldots \text{(i)}$$



Similarly

$$\angle ORS + \angle ORQ = C+D-90^\circ \quad \ldots \text{(ii)}$$

$$\therefore \quad \angle SPQ + \angle SRQ = 180^\circ.$$

∴ The points P, Q, R, S are concyclic.

Again, in the quadrilateral QCDQ′,

$$\angle DQ'Q = 270^\circ - C - D,$$

as $\angle CQQ' = 90^\circ$. ... (iii)

$$\therefore \quad \angle SQ'Q + \angle SRQ = 180^\circ, \text{ from (ii) and (iii)}.$$

Thus the points Q, R, S, Q′ are concyclic. That is, the circle PQRS passes through Q′. Similarly, it can be shown that it passes through P′, R′, and S′.

Question 1226.

(B. B. BAGI) :—ABC, DEF are six points on a conic. DE, DF meet BC in L_1, L_2; EF, ED meet CA in M_1, M_2; and FD, FE meet AB in N_1, N_2; then if AD, BE, CF meet in a point O, then L_1N_2, N_1M_2, M_1L_2 meet in the same point O.

Additional Solution by M. V. Ramakrishnan.

We have L_1ON_2 is the Pascal line of the hexagon EDABCF

N_1OM_2 „ „ „ FDEBAC;

and M_1OL_2 „ „ „ CBEFDA;

Hence the three lines L_1N_2, N_1M_2, M_1L_2 meet in the point O.

It may also be remarked that, since the Δs ABC, DEF are inscribed in a conic, their six sides touch a conic, so that O is the Brianchon point of the hexagon $L_1L_2\ N_1N_2\ M_1M_2$.

Question 1253.

(K. J. SAÑJAÑA, M.A. AND K. C. SHAH, M.A.) :—The centre of a conic described about the triangle of reference is at the point, whose trilinear co-ordinates are (f, g, h); prove that the lengths (r_1, r_2) of its principal semi-axes are given by the equation

$$4\,\Delta^2 r^4 - abcfgh\, r^2 \left\{ \frac{a^2}{\Delta - af} + \frac{b^2}{\Delta - bg} + \frac{c^2}{\Delta - ch} \right\}$$
$$+ \frac{a^2 b^2 c^2 f^2 g^2 h^2\, \Delta}{(\Delta - af)(\Delta - bg)(\Delta - ch)} = 0,$$

where a, b, c, Δ have their usual meanings.

Solutions: (1) *by M. V. Ramakrishnan, V. A. Mahalingam and*
(2) *by K. Satyanarayana.*

The lengths of the semi-axes of a conic, with centre at (x_0, y_0, z_0), whose equation in areal co-ordinates is

$$\phi\,(x, y, z) = ux^2 + vy^2 + wz^2 + 2fyz + 2\,gzx + 2\,hxy = 0,$$

are the roots of

$$Kr^4 + \theta r^2\, \phi\,(x_0\, y_0\, z_0) + 4\,\Delta^2\, \{\phi\,(x_0\, y_0\, z_0)\}^2 = 0;$$

where,
$$K = U + V + W + 2F + 2\,G + 2H,$$
$$\theta = \Sigma a^2 u - 2\,\Sigma\, f\, bc \cos A,$$

U, V, W, F, G, H being the minors of u, v, w, f, g, h in the determinant

$$\begin{vmatrix} u & h & g \\ h & v & f \\ g & f & w \end{vmatrix}$$

[A. C. JONES' *Analytical Conics.*]

Now, in the present case, it can be easily proved that the areal equation of the conic is,

$$\phi\,(xyz) = \Sigma 2\,af\,(\Delta - af)\, yz = 0,$$

where $af + bg + ch = 2\Delta$.

The areal co-ordinates of the centre of the conic are

$$\left[\frac{af}{2\Delta}, \frac{bg}{2\Delta}, \frac{ch}{2\Delta}\right]$$

Hence $K = -\Sigma a^2 f^2 (\Delta - af)^2 + 2\Sigma\, bgch\, (\Delta - bg)(\Delta - ch)$

$$= \frac{\Delta}{2}\,\Pi\,(bg + ch - af), \text{ on factorisation,}$$

$$= 4\Delta\,\Pi\,(\Delta - af);$$

$$\theta \quad = -2\,\Sigma\, abc\, f\,(\Delta - af)\cos A.$$

$$= -\tfrac{1}{2}\,\Sigma\, af\,(b^2 + c^2 - a^2)\,\Pi\, bg + ch - af)$$

$$= -2\,\Sigma\, a^2\,(\Delta - bg)(\Delta - ch);$$

$$\phi\,(x_0\, y_0\, z_0) = \frac{2\, afbgh}{4\,\Delta^2}\;\Sigma\;(\Delta - af) = \frac{abcfgh}{2\Delta}.$$

Hence the equation giving the lengths ot the axes reduces to

$$4\,\Delta^2 r^2 - abcfgh\, r^2\left\{\Sigma\frac{a^2}{\Delta - af}\right\} + \frac{a^2b^2c^2f^2g^2h^2\,\Delta}{\Pi\,(\Delta - af)} = 0.$$

(2) Let O be the centre of the conic, OD the perpendicular on BC. M the mid-pt. of BC; and let OL, AK be parallel to OD, OM; OX, OY parallel to BC, AK; OMD $= \omega$.

It is easily seen that

$$\left.\begin{aligned} OM &= f \operatorname{cosec}\omega,\ DM = \frac{a}{2} - h\operatorname{cosec} B - f\cot B \\ AK &= \frac{2\Delta}{a}\operatorname{cosec}\omega,\ LK = \frac{c}{f}\left[\frac{a}{2}\sin B - h\, f\cos B\right] \end{aligned}\right\} \quad \dots \quad (1)$$

$$\therefore \quad BK = \frac{\Delta - ch}{f},\ CK = \frac{\Delta - bg}{f}.$$

$$\therefore \quad a.\, K^2B = \left(\frac{\Delta - ba}{f}\right)c^2 + \left(\frac{\Delta - ch}{f}\right)b^2 - a\frac{(\Delta - bg)(\Delta - ch)}{f^2};$$

or using the value of AK trom (1)

$$4\Delta^2 f^2 \operatorname{cosec}^2\omega = -a^2(\Delta - bg)(\Delta - ch) + af\,[c^2(\Delta - bg) + b^2(\Delta - ch)]. \quad \dots \quad (2)$$

Referring to OX, OY, the equation of the conic is

$$a'x^2 + b'y^2 = 1;$$

the co-ordinates of A are $\left[\frac{bg-ch}{af}\right]$, $\left[\frac{a}{2}\frac{bg+ch}{af} f \operatorname{cosec} \omega.\right]$

.... B are $-\frac{a}{2}$, $-f \operatorname{cosec} \omega$.

.... C are $\frac{a}{2}$, $-f \operatorname{cosec} \omega$.

Hence

$$\left.\begin{aligned} a'\left(\frac{a}{2}\right)^2 + b'(f \operatorname{cosec} \omega)^2 &= 1, \\ a'\left(\frac{a^2}{2}\right)\left(\frac{bg-ch}{af}\right)^2 + b'(f \operatorname{cosec} \omega)^2\left(\frac{bg+ch}{af}\right)^2 &= 1. \end{aligned}\right\} \ldots (3)$$

$$\therefore \quad a'\left(\frac{a}{2}\right)^2 4(\Delta - bg)(\Delta - ch) = b'.\, 4(f \operatorname{cosec} \omega)^2 \Delta(\Delta - af).$$

$$\therefore \quad \frac{a'}{4(f \operatorname{cosec} \omega)^2 \Delta(\Delta - af)} = \frac{b'}{a^2(\Delta - bg)(\Delta - ch)}$$

$$= \frac{1}{(af \operatorname{cosec} \omega)^2 [\Delta(\Delta - af) + (\Delta - bg)(\Delta - ch)]}$$

$$= \frac{1}{abcfgh \cdot af \operatorname{cosec}^2 \omega}$$

$$= \frac{a' + b'}{a^2(\Delta = bg)(\Delta - ch) + 4f^2 \operatorname{cosec}^2 \omega\, \Delta(\Delta - af)}$$

$$= \frac{a' + b'}{\frac{af}{\Delta}.\ \sum\left[a^2(\Delta - bg).\ (\Delta - ch)\right]}$$

$$= \frac{(a'b')^{\frac{1}{2}}}{2af \operatorname{cosec} \omega\, [\Delta(\Delta - af)(\Delta - bg)(\Delta - ch)]^{\frac{1}{2}}},$$

by using (2) and (3).

$$\therefore \frac{1}{r_1^2} + \frac{1}{r_2^2} = (a' + b') \operatorname{cosec}^2 \omega = \frac{[\Sigma a^2(\Delta - bg)(\Delta - ch)]}{abcfgh \,.\, \Delta}.$$

$$\frac{1}{r_1^2 r_2^2} = a'b' \operatorname{cosec}^2 \omega = \frac{4\Delta(\Delta - af)(\Delta - bg)(\Delta - ch)}{a^2b^2c^2f^2g^2h^2}.$$

Hence the result.

Questions for Solution.

1303. (V. THIRUVENKATACHARI) :—Show that

$$\int_0^\infty \left(\frac{e^{-t}}{t(e^t+1)} - \frac{e^{-t}}{e^{2t}-1}\right) dt = \tfrac{1}{2}\log 4\pi + 1 + \tfrac{1}{2}C,$$

where C is Euler's constant.

1304. (N. R. JAYA RAO, M.A.) :—If

$$\frac{\pi}{2}\cdot\frac{1.3.5.7\ldots(2k-1)}{2.4.6.8\ldots\quad 2k} = f(k),$$

then, show that

$$\int_0^\infty \frac{\sin^{2k+1} x}{x}\, dx = f(k);$$

and

$$\int_0^\infty \frac{\sin^{2k} x}{x}\, dx = f(k-1).$$

1305. (V. RAMASWAMI AIYAR) :—By the "Frame of Similitude" of three directly similar figures let us mean the figure formed by their double points and their director point—to which may be added the circle of similitude, the invariable points, the annex triangles and circles, and so on.

Given three directly similar figures, prove that their 'frame of similitude' remains unaltered in form, if they are subjected to any relative motions of translation.

If two of the figures be fixed and the third move without rotation so that the frame of similitude remains unaltered in size also, show that every point of the third figure describes a circle.

1306. (N. B. MITRA) :—Prove that if p is a prime, then the period of $1/p$ (when converted into a recurring decimal) will run to an *even* submultiple of $(p-1)$ figures, if and only if $p \equiv \pm 3^n$, mod. 40, where n is any positive integer or 0; otherwise it will run to its full period of $p-1$ figures or to some *odd* submultiple of $(p-1)$ figures. $(p \neq 2$ or $5)$.

1307. (R. GOPALASWAMI) :—Prove that the Lemoine point and the Euler line are pole and polar with respect to the rectangular hyperbola through the vertices and the centroid.

1308. (T. VIJIARAGHAVAN) :—If N contains no prime factor of the form $(4n - 1)$ and is not a multiple of 4, then, the number of ways of representing N as the sum of 2 mutually prime squares is 2^{r-1} where r is the number of *different* prime factors of the form $(4n + 1)$ of N. Hence derive the well-known formula for the number of ways of representing a number as the sum of 2 squares.

1309. (S. N. KUMARASWAMY) :—In any triangle ABC draw a straight line AD to the side BC such that the radii of the in-circles of ABD, ACD may be equal. Explain any ambiguity in the solution.

1310. (V. V. SATHYANARAYANA MURTHY) :—Prove that the axes of a conic, inscribed in a fixed quadrilateral, whose sides touch a circle, envelope a parabola, whose directrix is the line through the mid. pts. of the diagonals of the quadrilateral.

1311. (K. SATYANARAYANA) :—AB is a fixed diameter of a rectangular hyperbola and P any point on the curve. Show that the locus of the nine-points centre of the triangle ABP is the diameter of the curve inclined to the axis at the same angle as AB.

1312. (B. B. BAGI) :—If OA_0A_1, $A_0A_1A_2$, $A_1A_2A_3$, etc. are similar triangles in a plane such that $A_0A_1 = k . OA$ $(k < 1)$. Show that if the process is continued indefinitely the point A_n tends to a definite point in the plane. Give a geometrical construction for it.

1313. (M. BHIMASENA RAO) :—Show how to find three integers whose sum is a square, and the sum of whose cubes a fourth power; *e.g.*, $12 + 17 + 20 = 7^2$; $12^3 + 17^3 + 20^3 = 11^4$.

1314. (A. A. KRISHNASWAMI IYENGAR) :—Prove that the sides and the medians of a triangle cannot *all* be rational.

1315. (K. J. SANJANA) :—The trilinear normal co-ordinates of a point P with regard to the triangle of reference ABC being proportional to sec (B—C), sec (C—A), sec (A—B), prove that

$$\cot PBC \cot PCA \cot PAB + \cot PCB \cot PBA \cot PAC$$

$$= \frac{12 \cot \omega + 20 \cot A \cot B \cot C}{2 + \cot A \cot B \cot C \cot \omega + \cot^2 A \cot^2 B \cot^2 C}.$$

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LIST OF JOURNALS RECEIVED BY THE SOCIETY.

1 Messenger of Mathematics
2 Quarterly Journal of Mathematics
3 Mathematical Gazette
4 The Annals of Mathematics
5 American Journal of Mathematics
6 Bulletin of the American Mathematical Society
7 Transactions of the American Mathematical Society
8 Monthly Notices of the Royal Astronomical Society
9 Proceedings of the Royal Society of London
10 The Philosophical Magazine and Journal of Science
11 Astrophysical Journal
12 Crelle's Journal
13 L'intermediaere des Mathematiciens
14 Mathematische Annalen
15 Philosophical Transactions of the Royal Society of London
16 Acta Mathematica
17 Popular Astronomy
18 Proceedings of the Edinburgh Mathematical Society
19 Proceedings of the London Mathematical Society
20 Mathematics Teacher
21 Bulletin of the Calcutta Mathematical Society
22 The Tohoku Mathematical Journal
23 Nature
24 The American Mathematical Monthly
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